PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Complete Set of Uniform Polyhedra

J. Skilling

Phil. Trans. R. Soc. Lond. A 1975 **278**, 111-135
doi: 10.1098/rsta.1975.0022

THE COMPLETE SET OF UNIFORM POLYHEDRA

By J. SKILLING

Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge

(*Communicated by M. S. Longuet-Higgins, F.R.S. – Received* 26 *March* 1974)

[Plate 13]

CONTENTS

A definitive enumeration of all uniform polyhedra is presented (a uniform polyhedron has all vertices equivalent and all its faces regular polygons). It is shown that the set of uniform polyhedra presented by Coxeter, Longuet-Higgins & Miller (1953) is essentially complete. However, after a natural extension of their definition of a polyhedron to allow the coincidence of two or more edges, one extra polyhedron is found, namely the great disnub dirhombidodecahedron.

1. Introduction

A *uniform polyhedron* has all its vertices equivalent, and all its faces must be regular plane polygons. Neither the faces nor the polyhedron need be convex, and different types of face may appear in the same polyhedron. The polyhedron must, however, be three dimensional, so that tessellations of a plane are excluded. Also excluded are cases where the polyhedron is a compound of other polyhedra.

The problem of listing *all* the possible uniform polyhedra has been with us for some time. Kepler was the first to list all the convex uniform polyhedra, but he also generalized the problem by introducing non-convex polyhedra. To date, the most detailed exposition of the problem has been the paper of Coxeter *et al.* (1953), in which are listed 75 uniform polyhedra, in addition to two denumerably infinite sets, and in which the authors express their belief that no more are to be found.

This present paper presents a systematic and complete enumeration by computer of all uniform polyhedra. I show that the list of Coxeter *et al.* is indeed complete as regards uniform polyhedra in which only *two* faces meet at any edge. The natural generalization that any *even* number of faces may meet at an edge allows just one extra polyhedron to be included in the set.

Now the vertices of a uniform polyhedron belong to one of the three dimensional symmetry groups, this being the definition of 'equivalent' vertices in the present context. Such vertices must lie on a sphere, known as the *circumsphere* of the polyhedron, centred on the centroid of the vertices. Uniform polyhedra may thus be classified by the symmetry group to which they belong. These groups are all either subgroups of the (maximal) *icosahedral* group or subgroups of the (maximal) *cubic* group or subgroups of one of the denumerably infinite groups with *prism* (i.e. dihedral) symmetry.

I shall first discuss, and eliminate, prism symmetry.

Prism symmetry

The full prism symmetry group D_{nh} of order n has a n-fold axis of rotation passing through the 'pole' of the circumsphere, n twofold axes of rotation round the 'equator', and n mirror planes (figure 1). The number n can be any positive integer, although $n = 1$ and $n = 2$ are subgroups of the cubic and icosahedral groups, so we take $n \geqslant 3$. There can be up to $4n$ separate vertices, arranged in two parallel planes either side of the 'equator'.

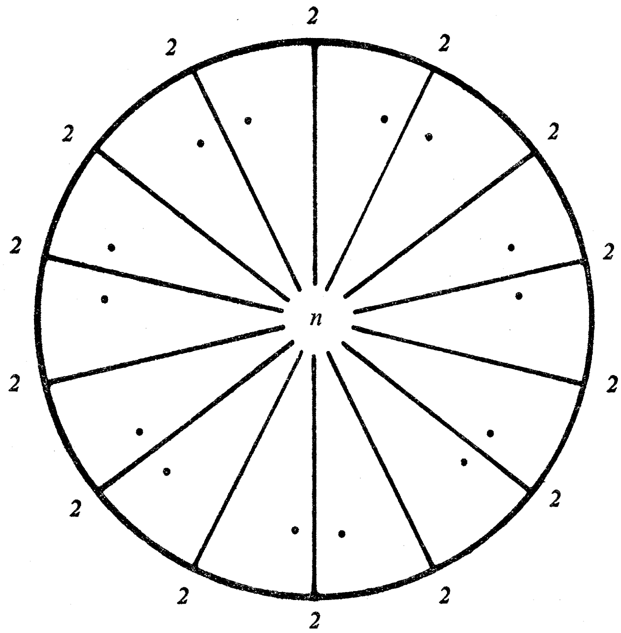

FIGURE 1. The prism symmetry group of order 7, in stereographic projection (upper hemisphere only).

Any face which connects vertices in both planes, being a regular polygon and hence cyclic, can have at most two of its vertices in either of the planes. Accordingly, such a face has at most 4 vertices, and must be a triangle or a square. It is then easy to see that the family of prisms (where the connecting faces are squares) and antiprisms (where the connecting faces are triangles) exhausts all the possibilities.

Hereafter prism symmetry will be ignored, as only icosahedral and cubic symmetries need be further considered.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Icosahedral and cubic symmetries

The maximal icosahedral group I_h has 120 elements, arranged in the pattern of fivefold, threefold and twofold axes shown in figure 2, each of which rotations can be combined with inversion. Explicitly, the operators are as follows.

(*a*) ***1***, the identity operator,

(*b*) $\boldsymbol{5}^1$, $\boldsymbol{5}^2$, $\boldsymbol{5}^{-2}$, $\boldsymbol{5}^{-1}$, denoting rotations of $+\frac{2}{5}\pi$, $+\frac{4}{5}\pi$, $-\frac{4}{5}\pi$, $-\frac{2}{5}\pi$ respectively about any of the six fivefold axes,

(*c*) $\boldsymbol{3}^1$, $\boldsymbol{3}^{-1}$, denoting rotations of $+\frac{2}{3}\pi$, $-\frac{2}{3}\pi$ respectively about any of the ten threefold axes,

(*d*) ***2***, a rotation of π about any of the fifteen twofold axes,

(*e*) any of the above 60 operators combined with inversion through the centre of the sphere.

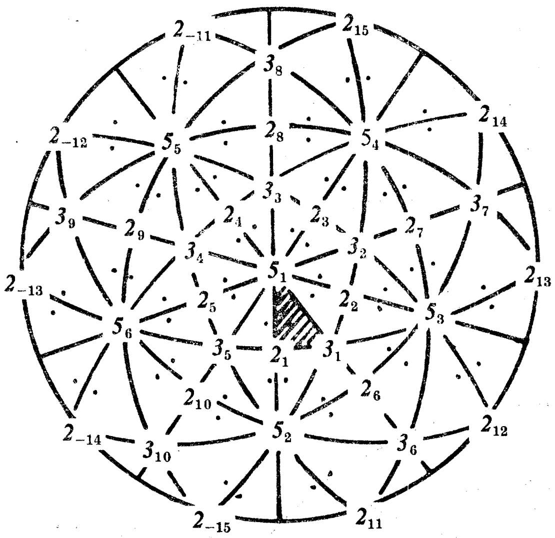

FIGURE 2. The maximal icosahedral group (upper hemisphere only), with principal Möbius triangle shaded.

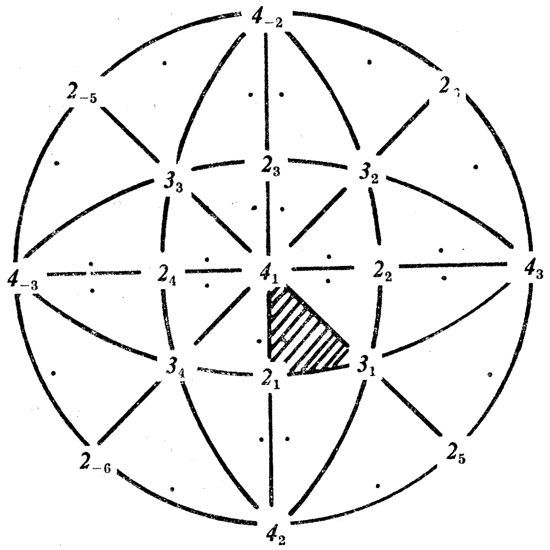

FIGURE 3. The maximal cubic group (upper hemisphere only), with principal Möbius triangle shaded.

The maximal cubic group O_h has the 48 elements shown in figure 3, which comprise

(*a*) ***1***, the identity operator,

(*b*) $\boldsymbol{4}^1$, $\boldsymbol{4}^2$, $\boldsymbol{4}^{-1}$, denoting rotations of $+\frac{1}{2}\pi$, π, $-\frac{1}{2}\pi$ respectively about any of the three fourfold axes,

(*c*) $\boldsymbol{3}^1$, $\boldsymbol{3}^{-1}$, rotations about any of the four threefold axes,

(*d*) ***2***, a rotation about any of the six twofold axes,

(*e*) any of the above combined with inversion.

Since the number of vertices cannot exceed 120, there is clearly only a finite number of ways of joining them with regular polygon faces to form a uniform polyhedron. The task is to reduce the number of possibilities from around factorial 120 to manageable proportions.

The technique I use is to build up the polyhedra in four stages.

First: calculate all possible sets of vertices and edge lengths;

secondly: link these with all possible edges;

thirdly: select any plane polygons which occur in the system of edges, then

fourthly: build a polyhedron from some or all of these faces.

2. Generation of polyhedron vertices and edge lengths

Consider the Cartesian coordinates $\boldsymbol{x} = (x, y, z)$ of some vertex P of the polyhedron we hope to build, together with the edge length s, as a quadruplet of numbers to be determined from 4 suitable conditions.

$$(x, y, z; s) \quad \text{are unknowns.}$$

The coordinates x, y, z are taken with respect to axes fixed in the symmetry group, with origin at the centre. If we take the *radius of the circumsphere to be unity*, then one condition is

$$x^2+y^2+z^2 = 1. \tag{1}$$

Three more conditions come from considerations of symmetry. First, we note that each of the polyhedron's vertices can be obtained from P by applying just one of the symmetry operators. In special cases, some of the vertices so defined may be coincident, but that does not affect the argument at this stage. Suppose P is connected to the vertex $\boldsymbol{R}(\boldsymbol{x})$ by a polyhedron edge of length s, where $\boldsymbol{R}$ is a rotation through an angle θ about an axis $\boldsymbol{n}$. Then (figure 4),

$$\tfrac{1}{2}s = [1-(\boldsymbol{x}\cdot\boldsymbol{n})^2]^{\frac{1}{2}} \sin(\tfrac{1}{2}\theta), \tag{2a}$$

given unit radius for the circumsphere. If $\boldsymbol{R}$ is an improper rotation, involving inversion, then this equation should instead read

$$\tfrac{1}{2}(4-s^2)^{\frac{1}{2}} = [1-(\boldsymbol{x}\cdot\boldsymbol{n})^2]^{\frac{1}{2}} \sin(\tfrac{1}{2}\theta). \tag{2b}$$

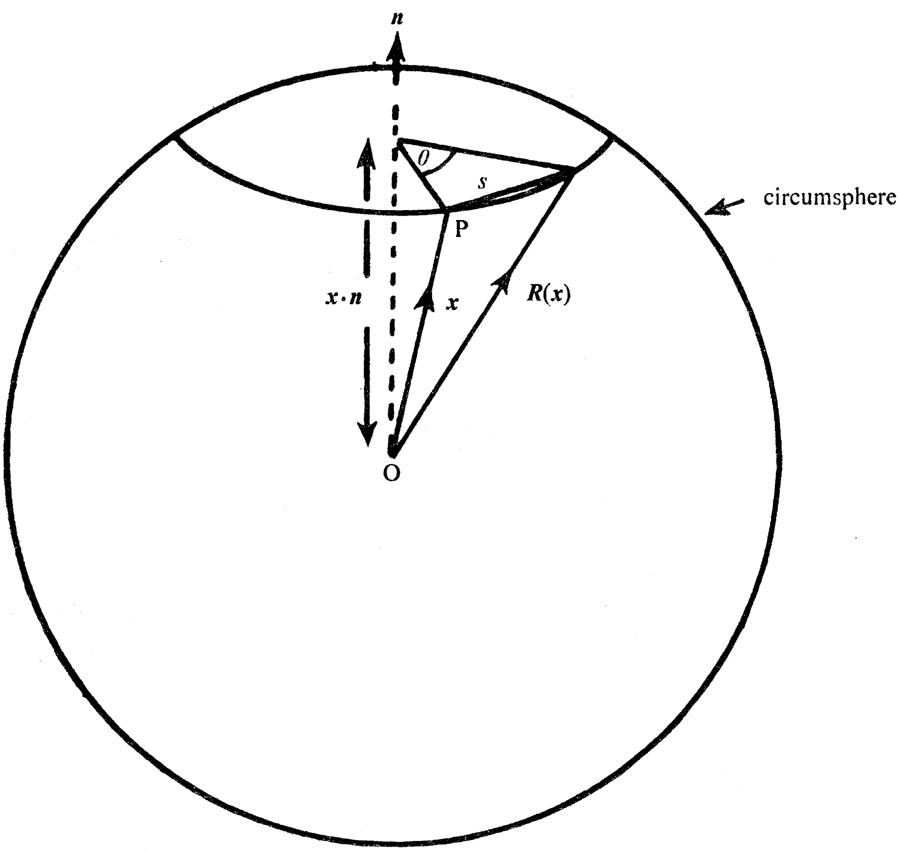

Figure 4. Edge lengths in terms of rotation $\boldsymbol{R}$ through θ about $\boldsymbol{n}$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

For later purposes, these are rewritten as

$$\boldsymbol{x}\cdot\boldsymbol{n} = \sqrt{\left[\frac{-\cos\theta}{1-\cos\theta} \pm \frac{1-\frac{1}{2}s^2}{1-\cos\theta}\right]}, \tag{3}$$

where the upper (lower) sign holds if $\boldsymbol{R}$ represents a proper (improper) rotation, and where the square root may have either sign.

Equation (3) is one condition on $(x, y, z; s)$ and we shall generate three such conditions by using different operators $\boldsymbol{R}$. At first sight, we might suppose that a polyhedron could be built up in which P was connected to other vertices by *only one* operator $\boldsymbol{R}$. (The operator $\boldsymbol{R}^{-1}$ gives the same condition (3) as does $\boldsymbol{R}$, so for the time being we do not distinguish between $\boldsymbol{R}$ and $\boldsymbol{R}^{-1}$.) However, this would not give a connected polyhedron. For example, if $\boldsymbol{R}$ were a rotation of $\frac{1}{2}\pi$ about a fourfold cubic axis, the polyhedron would consist of disconnected squares (figure 5).

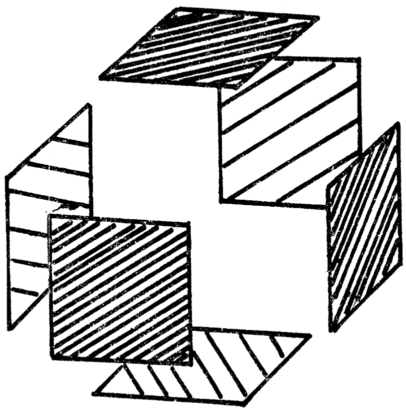

FIGURE 5. Invalid polyhedron of disconnected squares.

Such disconnected faces can only be joined up if vertices are allowed to coalesce. But coalescence implies that there exists $\boldsymbol{R}'$ (not the identity) such that $\boldsymbol{R}'(\boldsymbol{x}) = \boldsymbol{x}$. In this case $\boldsymbol{R}(\boldsymbol{x}) = \boldsymbol{R}[\boldsymbol{R}'(\boldsymbol{x})]$, so that the operator $(\boldsymbol{R}\boldsymbol{R}')$, itself an element of the group, would also satisfy the condition (3). Thus, for an acceptable polyhedron, P must be connected to other vertices by more than one operator. Polyhedra do exist in which P is connected to other vertices by exactly two independent operators, but this is a tricky case which is discussed in the appendix.

All more complex polyhedra have the chosen vertex P connected to other vertices by *at least three* different operators $\boldsymbol{R}$ (we continue to ignore the distinction between $\boldsymbol{R}$ and $\boldsymbol{R}^{-1}$). Each operator gives a condition like equation (3), so we can proceed to enumerate all possible vertices of all the more complex polyhedra by listing all possible combinations of *three* operators, and solving equations (1) and (3). Of course, any polyhedron in which P is connected to other vertices by more than three operators will enter the enumeration more than once. The lists of independent combinations of three operators which I used, and in which there may be some remaining redundancy, comprised 894 combinations of icosahedral operators and 186 combinations of cubic operators. The procedure described in the appendix for dealing with the case in which only two operators of the group satisfy equation (3) results in adding 28 operator combinations to the icosahedral list and 12 to the cubic. Thus only of the order of a thousand cases need be considered in order to enumerate all uniform polyhedra.

Computations of $(x, y, z; s)$

For each choice of three operators, equation (1) and the three equations (3) were solved in the following way. First, x, y, z were determined in terms of s by inverting the matrix of components of the vectors $\boldsymbol{n}$ which appears on the left of the three equations (3). Then x, y, z were substituted into equation (1)

$$x^2+y^2+z^2 = 1$$

to yield an expression involving s as the only unknown. When the square root operations are removed, it takes the form of a quartic equation for $(1-\frac{1}{2}s^2)$

$$Q(1-\tfrac{1}{2}s^2) = 0. \tag{4}$$

The computer program for finding $(x, y, z; s)$ starts by using the numerical values of θ and $\boldsymbol{n}$ associated with the chosen operators to determine the quartic Q. In the special cases where the rotation axes are coplanar or colinear, or where two or more rotations are of equal magnitude, the quartic factorizes into quadratics, or even further. These cases were detected by the program, which took appropriate action. However, the quartic generally had to be solved directly, with due care taken of multiple roots. Ignoring values of $(1-\frac{1}{2}s^2)$ outside the admissible range $0 < s < 2$, the remaining (real) values were used with each possible combination of square root signs in (3) to determine possible values of x, y, z. Of these, the program only accepted real values of x, y, z which obeyed equation (1). This procedure resulted in a list of all acceptable solutions $(x, y, z; s)$ for a given choice of three operators.

Rounding errors (checked to be proportional to the arithmetic accuracy) were a considerable nuisance in this part of the computation, causing a loss of up to 6 decimal places. In the final version of the program, the arithmetic was carried out to an ostensible accuracy of 23 decimal places, to give results accurate to at least 17 places. Branch decisions in the program were taken at the plausibly safe level of 15 places. Nevertheless, the generated quadruplets were accepted by the program if they satisfied the equations (1) and (3) at the coarse level of only 4 decimal places. While discriminating against clearly invalid solutions (which arise because of the ambiguous square root signs), this coarse level would accept even the highly inaccurate solutions which might be produced by an undetected multiple root. No such cases were found, as all quadruplets accepted on the basis of 4 decimal places were in fact accurate to at least 17 places, and the program seems always to have worked correctly.

After rotating each vertex (x, y, z) into some 'principal' Möbius triangle† (figures 2, 3) the output lists contained 308 different quadruplets in the icosahedral case, and 68 in the cubic case. One vertex from every uniform polyhedron must appear in these lists.

3. Generation of polyhedron edges

From one particular quadruplet $(x, y, z; s)$ the symmetry group will generate a vertex list having 120 (icosahedral) or 48 (cubic) entries, or half this number if (x, y, z) lies on a symmetry plane. The specially simple cases in which (x, y, z) lies on a symmetry axis were in fact extracted from the lists of quadruplets, and treated by hand.

† A *Möbius triangle* is a spherical triangle such that the set of triangles formed by applying the symmetry group to it exactly covers the sphere.

Those vertices which lie at the correct distance s from $\boldsymbol{x}$ are candidates for appearing in the vertex figure† of the final polyhedron, so the program continues by sublisting these vertices and the operators which produce them. For example, the *snub cube* (figure 6) is known to have the vertex figure shown in figure 7. Starting from the quadruplet appropriate to the snub cube, the program finds that five operators, namely $\boldsymbol{4}^1_1$, $\boldsymbol{4}^{-1}_1$, $\boldsymbol{3}^1_1$, $\boldsymbol{3}^{-1}_1$, $\boldsymbol{2}_1$ give vertices distant s from $\boldsymbol{x}$, and could give edges of the polyhedron (figure 8).

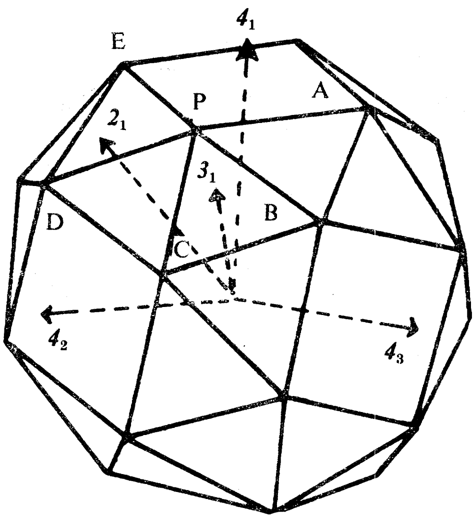

FIGURE 6. The snub cube.

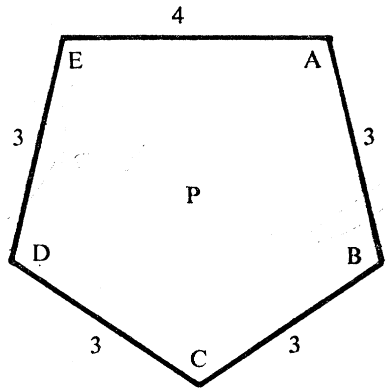

FIGURE 7. Vertex figure of snub cube.

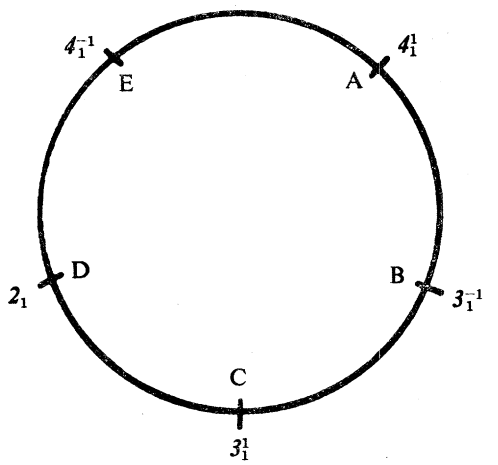

FIGURE 8. Candidates for vertex figure of vertex P of snub cube.

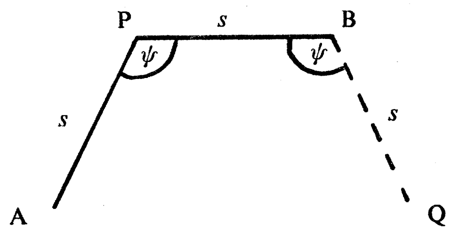

FIGURE 9. Continuation of face APB-- to Q.

4. GENERATION OF POLYHEDRON FACES

The program now tries to build a planar *face* from each pair of candidate vertex figure points in turn. Considering again the snub cube, take the pair of vertices $\boldsymbol{4}^1_1(\boldsymbol{x})$, $\boldsymbol{3}^{-1}_1(\boldsymbol{x})$ as an example (A and B of figure 8). The three successive vertices A, P, B, where P is the vertex $\boldsymbol{x}$, determine the

† The *vertex figure* of P is the (cyclic) polygon connecting those vertices of the polyhedron which are directly joined to P, and whose edges lie in the faces of the polyhedron. Uniform polyhedra may be specified by the vertex figure of any of their vertices.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

candidate face completely. The program now constructs the *next* vertex of the face, Q say, following A, P, B (figure 9). Then P and Q must be linked together in the vertex figure of B. Equivalently, C and D must be linked together in the vertex figure of P. Continuing in this way round the candidate face, D and E must also be linked in the vertex figure of P, and then the process returns to the original pair of vertex points, A and B. The program has, in fact, found the snub triangle faces (figure 10*a*).

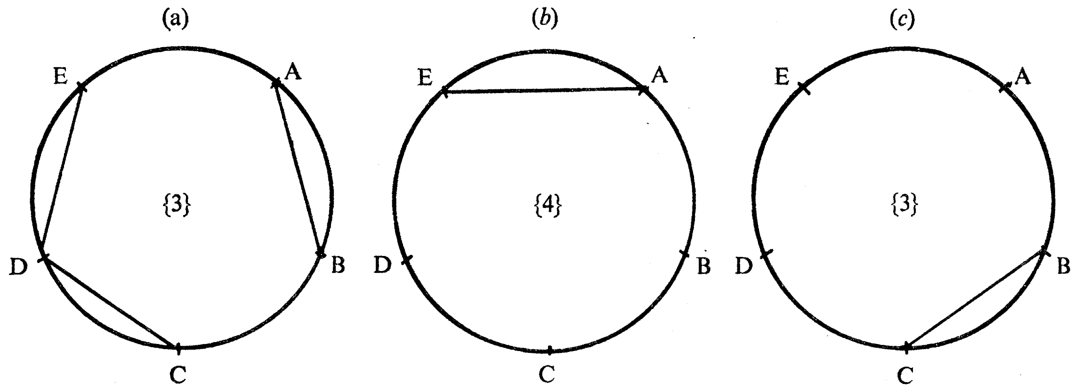

FIGURE 10. Allowed links between vertex points of P.

Of the other pairs of vertices which are still to be tried, only AE and BC turn out to give valid polygons. The pair (A, E) gives the square faces (figure 10*b*), and the pair (B, C) gives the non-snub triangles (figure 10*c*). All the possible faces among the allowed edges have now been found.

5. GENERATION OF COMPLETE POLYHEDRA

It is still necessary to combine these sets of plane polygons to form a complete polyhedron, having an even number of faces meeting at every edge. Accordingly, the program inspects all the possible combinations of faces, and plots the vertex figures of successful combinations. In the snub cube example, the only successful combination is that which includes all the allowed faces. These plots, together with printed diagnostics, comprise the final output of the computer programs. There were generated in all, 115 icosahedral and 56 cubic vertex figures.

Polyhedra with fewer vertices than given by the maximal symmetry group nevertheless appear in the output lists, because in such cases the maximal group merely produces a compound of the appropriate number of less symmetrical polyhedra. One of the polyhedra will have vertices derived from x by applying the subgroup representing lower symmetry: the others will have vertices derived from the cosets of that subgroup. As long as the 120 (icosahedral) or 48 (cubic) vertices are all distinct, these less symmetrical polyhedra will be completely disjoint.

A complication arises if the vertices lie on symmetry planes, because all the 60 (icosahedral) or 24 (cubic) vertices can then be derived by applying an appropriate *sub*-group to x. In the icosahedral case, the only appropriate subgroup is the group I of icosahedral proper rotations. In the cubic case, there are two appropriate subgroups: if the vertices lie on symmetry planes which do not (do) contain threefold rotation axes, the two subgroups are (i) the group O of cubic proper rotations and (ii) the group T_d (T_h) of tetrahedral proper rotations together with those mirror planes which do (do not) contain threefold rotation axes. The program uses these subgroups in place of the maximal group to generate edges, faces, and finally complete polyhedra. Polyhedra of yet lower symmetry still appear as disjoint compounds. Of course, there is nothing to prevent the final polyhedron from having the symmetry of the maximal group, and in fact most of the polyhedra whose vertices lie on symmetry planes do exhibit maximal symmetry. However maximal symmetry is not required. Indeed, the great snub dodecicosidodecahedron (Wenninger 1971, p. 183) $|3\,\frac{5}{3}\,\frac{5}{2}$, whose 60 vertices lie on icosahedral symmetry planes, is an example of a polyhedron which nevertheless lacks mirror symmetry. By using the subgroup *I* of proper rotations, the program produces both enantiomorphs of its vertex figure (see figure 14*a*), while the compound with maximal symmetry is not disjoint and is in fact excluded by having three faces meeting at some of its edges.

The cases in which the vertices lie on symmetry axes were treated using all relevant symmetry subgroups.

Sometimes a vertex figure itself represents a compound. For example, the compound of an icosahedron and a great dodecahedron (figure 11) is admitted by my criteria because it contains an even number (four) of faces per edge, although inspection immediately reveals its compound nature.

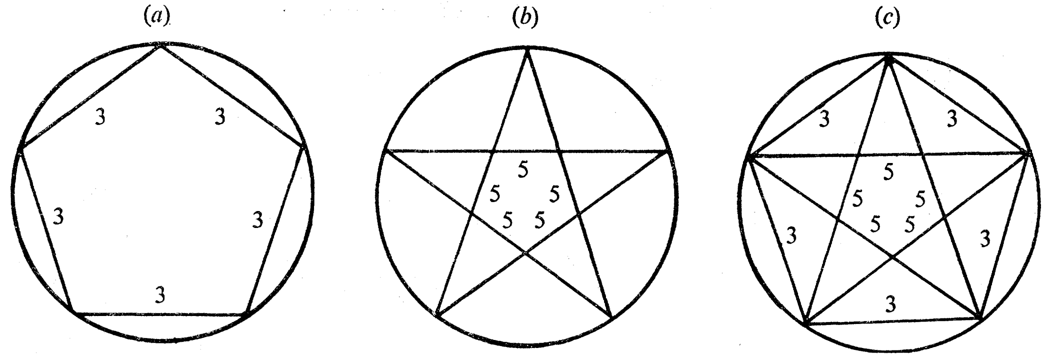

FIGURE 11. (*a*) Vertex figure of icosahedron. (*b*) Vertex figure of great dodecahedron. (*c*) Compound of both.

When repetitions and compounds are removed, the remaining list of polyhedra comprises exactly those uniform polyhedra listed by Coxeter, Longuet-Higgins & Miller in 1953 (excluding of course prisms and antiprisms), with just one addition. It is perhaps worth pointing out that my programs produced this agreement at once. Only after I judged the programs to be free of error did I check the results against Coxeter *et al.*, finding perfect agreement immediately.† Thus my enumeration is fully independent of that of Coxeter *et al.*

6. The great disnub dirhombidodecahedron

Only one uniform polyhedron exists which is not included in the paper by Coxeter *et al.*: it is special in having four faces meeting at some of its edges.

This polyhedron has 60 vertices lying on symmetry planes of the icosahedral group. All eight candidate vertices (figure 12) are included in the vertex figure. Actually, seven types of face

† A small internal error in that paper does not affect its conclusions. At the bottom of page 409, '$(5\,\frac{5}{2}\,\frac{5}{2})$' should be '$(3\,3\,\frac{4}{3})$, $(4\,4\,\frac{4}{3})$', and in the first display on page 410, '$(2\,\frac{5}{2}\,10)$' should be '$(2\,3\,\frac{8}{3})$, $(2\,4\,\frac{8}{3})$', which gives a list with *six* members instead of five.

(figure 13) are found among this system of edges, from which the programs built seven possible polyhedra. Six of these polyhedra (figure 14) are conventional in having only two faces meeting at each edge. The seventh combination of faces (figure 15) is the new polyhedron.

This polyhedron has 204 faces (easily a record among uniform polyhedra), comprising 12 coplanar pairs of pentagrams, 30 coplanar pairs of snub squares, and 120 non-coplanar triangles arranged in two distinct snub sets. It is unique in having both snub squares and snub triangles. Although twelve faces meet at each vertex, there are only eight edges to a vertex,

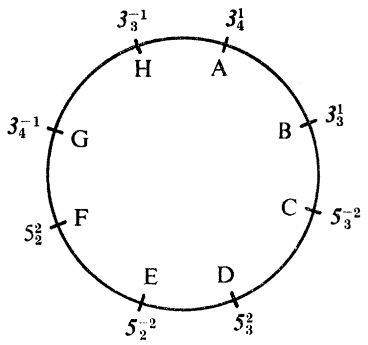

FIGURE 12. Candidate vertices for vertex figure.

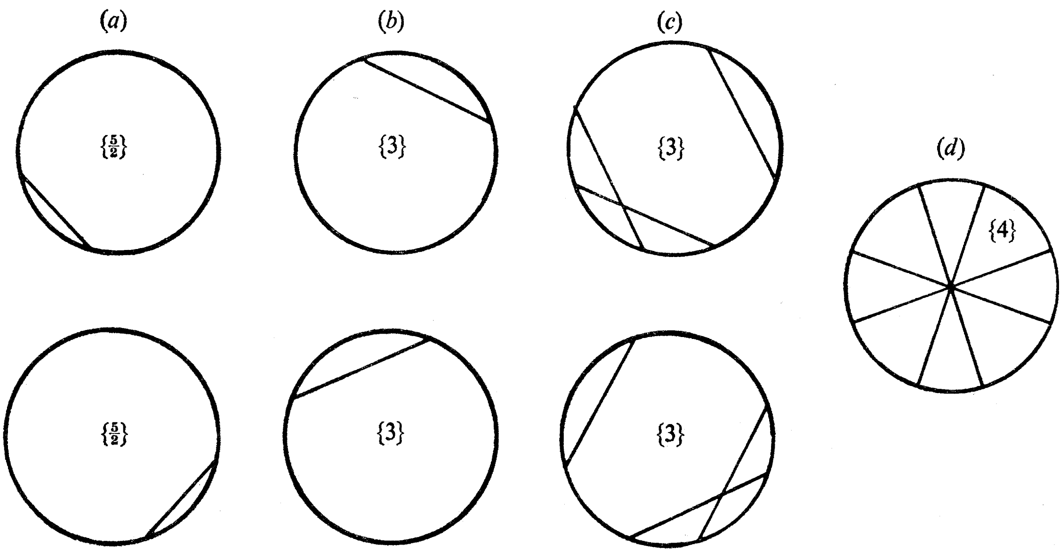

FIGURE 13. Candidate sets of faces: (*a*) Enantiomorphous sets of pentagrams (12 in each set). (*b*) Enantiomorphous sets of triangles (20 in each set). (*c*) Enantiomorphous sets of snub triangles (60 in each set). (*d*) Snub squares (60 in all).

because half the edges are 'double', connecting four faces at once. It is not a compound. Although its vertex figure contains that of a tetrahemihexahedron (Wenninger 1971, p. 101) $\frac{3}{2}\ 3|2$ (i.e. two {3} connected by crossed {4}), the polyhedron as a whole does not contain a complete tetrahemihexahedron. The new polyhedron is closely related to the great dirhombicosidodecahedron (Wenninger 1971, p. 200) $|\frac{3}{2}\ \frac{5}{3}\ 3\ \frac{5}{2}$, which has the same vertices, edges, pentagrams and snub squares. Indeed one can derive the new polyhedron from it by replacing all the latter's (non-snub) triangles (figure 13*b*) by the sets of snub triangles (figure 13*c*). This suggests a way of traversing the vertex figure of the new polyhedron, namely AEFBDFHDCGECA (analogous to

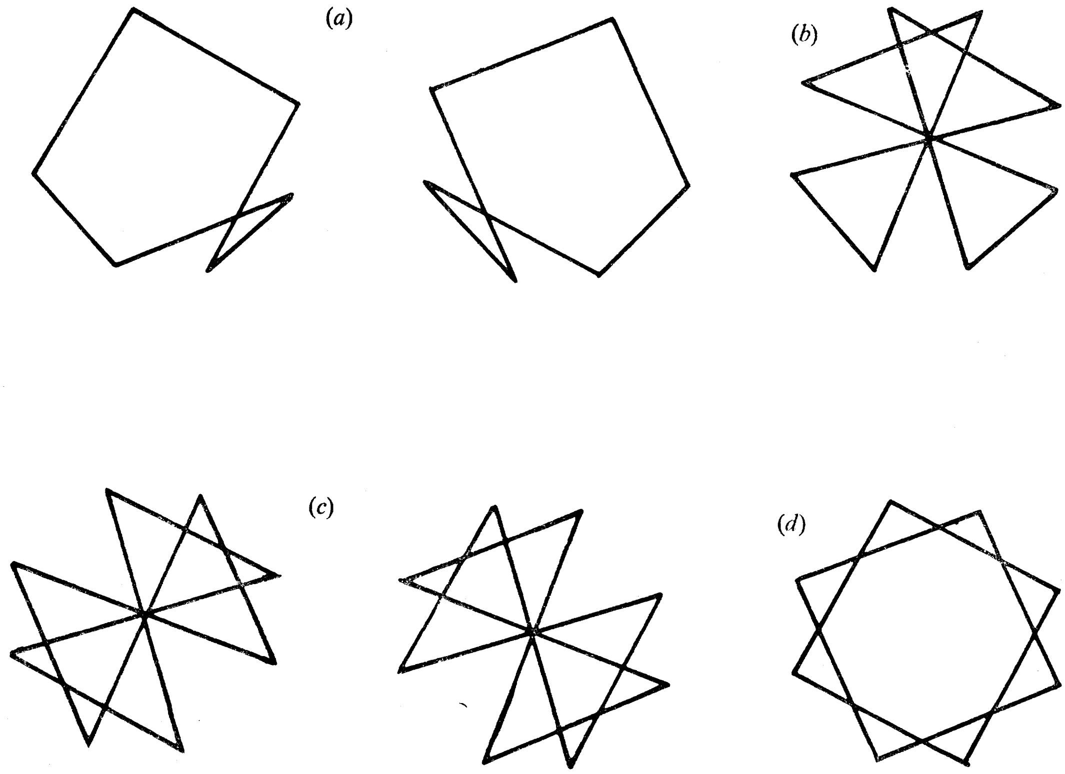

FIGURE 14. (*a*) Enantiomorphs of | 3 $\frac{5}{3}$ $\frac{5}{2}$. (*b*) | $\frac{3}{2}$ $\frac{5}{3}$ 3 $\frac{5}{2}$. (*c*) Enantiomorphous compounds of $\frac{3}{2}$ 3|2. (*d*) Compound of octahedra.

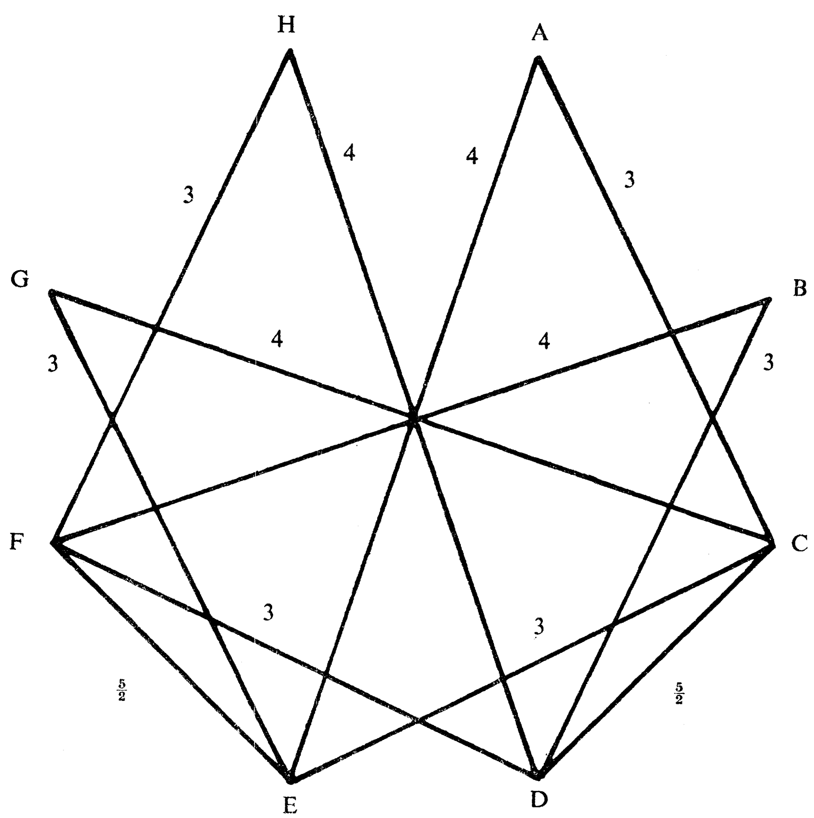

FIGURE 15. Vertex figure of great disnub dirhombidodecahedron, | ($\frac{3}{2}$) $\frac{5}{3}$ (3) $\frac{5}{2}$.

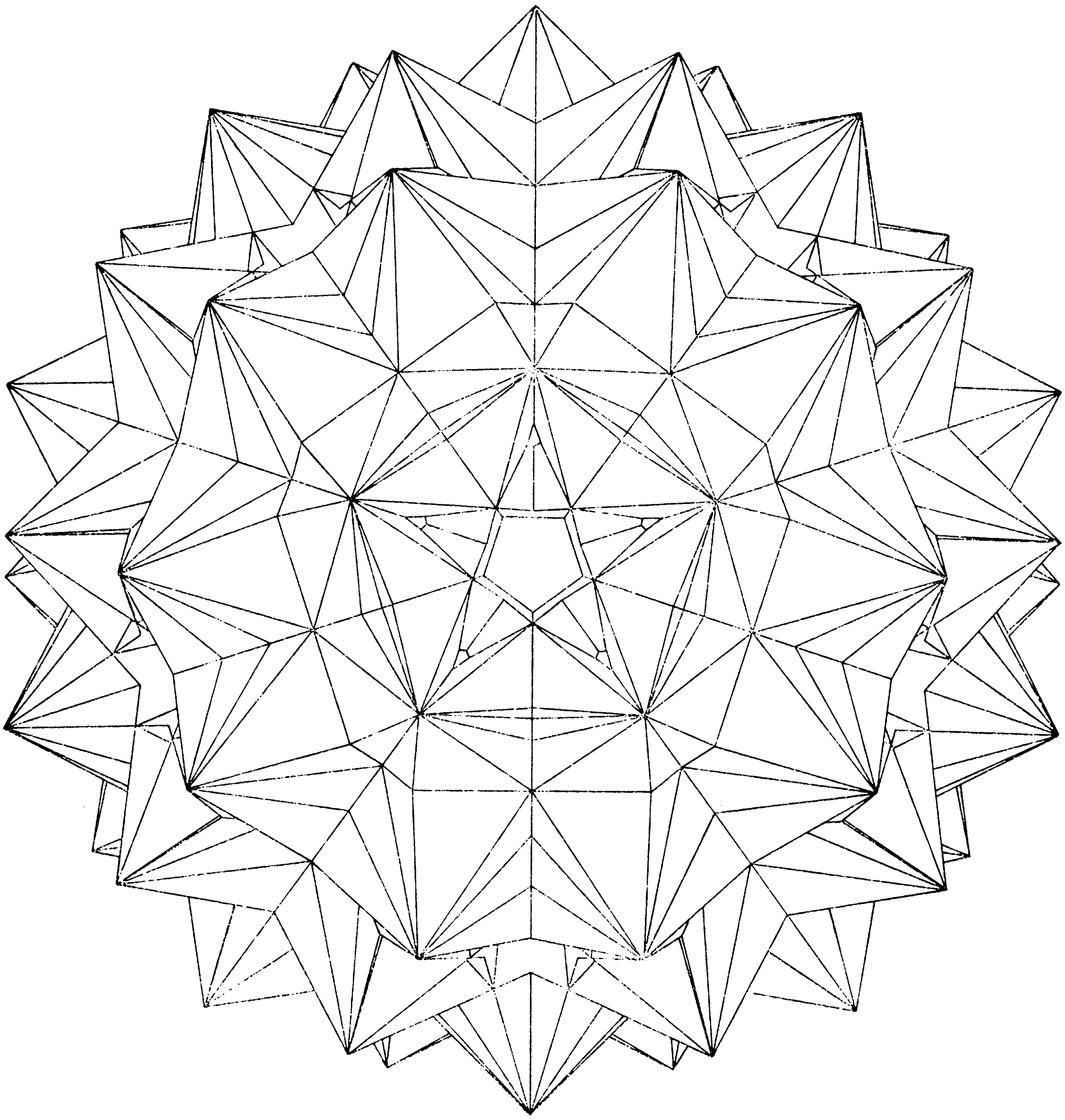

FIGURE 16. Line drawing of great disnub dirhombidodecahedron, $|\ (\frac{3}{2})\ \frac{5}{3}\ (3)\ \frac{5}{2}$.

TABLE 1. THE GREAT DISNUB DIRHOMBIDODECAHEDRON

symbol	$|\ (\frac{3}{2})\ \frac{5}{3}\ (3)\ \frac{5}{2}$
vertices	60
edges	240, of which 120 are double 8 edges meet at each vertex
faces	$24\{\frac{5}{2}\}+60\{4\}+120\{3\} = 204$ 12 faces meet at each vertex
edge length	$\surd 2$ circumradii
symmetry	maximal icosahedral
vertex figure	displayed in figure 15

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

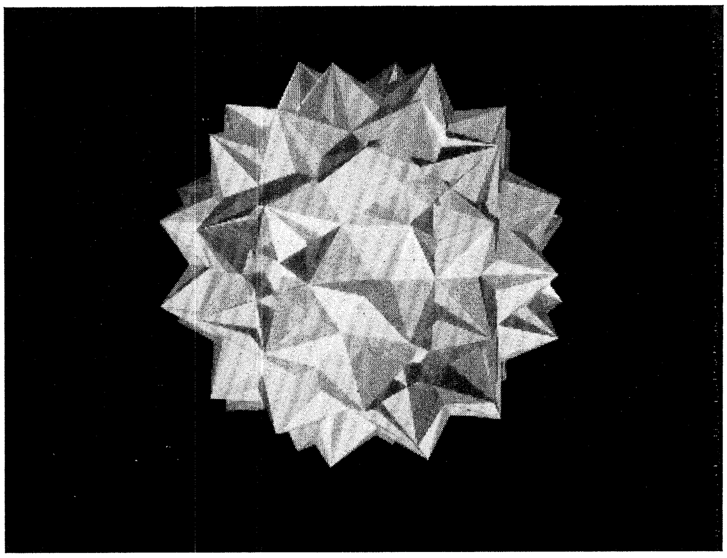

FIGURE 17. Computer photograph of great disnub dirhombidodecahedron.

(*Facing p.* 122)

AEFBHDCGA round the vertex figure of the great dirhombicosidodecahedron $|\frac{3}{2}\,\frac{5}{3}\,3\,\frac{5}{2}$). We can consider the edges where four faces meet as *double* edges comprising two colinear (hence coincident) edges, rather analogous to the coplanar (and partly coincident) faces which appear in this and other uniform polyhedra. Splitting the double edges as suggested above yields a single-sided surface for the new polyhedron. The density of the new polyhedron is indeterminate, since the snub squares pass through its centre.

A line drawing of the new polyhedron is exhibited in figure 16, and figure 17, plate 13 shows a computer-generated photograph from another viewpoint.†

Nomenclature

The new polyhedron's lack of relation to Wythoff's construction means that there is no obvious symbol for it clearly different from the symbol $|\frac{3}{2}\,\frac{5}{3}\,3\,\frac{5}{2}$ belonging to the great dirhombicosidodecahedron. It may be best to give it a symbol based on its snub squares, and I suggest

$$\text{`}|\,(\tfrac{3}{2})\,\tfrac{5}{3}\,(3)\,\tfrac{5}{2}\text{'}$$

where the parentheses are reminders to use *snub* triangles in the polyhedron.

The constituent faces suggest that the polyhedron be called the 'great disnub dirhombidodecahedron'. This is in keeping with the nomenclature used by Wenninger (1971) for the uniform polyhedra.

I am indebted to Dr J. C. P. Miller of the Cambridge University Computer Laboratory for introducing me to the study of uniform polyhedra. The line drawing (figure 16) was produced by Mr A. C. Norman, also of the Computer Laboratory, and the photograph was generated by Mr R. G. Newell of the Computer-Aided Design Centre, Cambridge. Professor M. S. Longuet-Higgins, F.R.S., of the Department of Applied Mathematics and Theoretical Physics, Cambridge, has made helpful comments on the manuscript. To all these people I extend my grateful thanks.

The computations were carried out on Cambridge University's TITAN computer.

References

Coxeter, H. S. M., Longuet-Higgins, M. S. & Miller, J. C. P. 1953 *Phil. Trans. R. Soc. Lond.* A **246**, 401–449.

Wenninger, M. J. 1971 *Polyhedron models*. Cambridge University Press.

Appendix. Case in which only two operators of the group satisfy equation (3)

The vertex figure of the point x can consist of at most four points $\boldsymbol{R}_1\,x$, $\boldsymbol{R}_1^{-1}\,x$, $\boldsymbol{R}_2\,x$, $\boldsymbol{R}_2^{-1}\,x$ where $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are the two operators. Since all vertex figures contain at least three points, the vertex figure must contain either four or three points.

Case 1

The vertex figure contains four points.

Neither $\boldsymbol{R}_1$ nor $\boldsymbol{R}_2$ may be a twofold operator, otherwise some of the four points in the vertex figure coalesce. Each point in the vertex figure must be joined to an even number (i.e. two) of

† Note added in preparation: I have now constructed a card model of the polyhedron which confirms its structure.

other included points, because this number gives the number of polyhedron faces meeting at an edge. There are three independent ways of thus joining the four points, disregarding cyclic permutation and overall reversal, namely $\boldsymbol{R}_1\boldsymbol{x}$ to $\boldsymbol{R}_1^{-1}\boldsymbol{x}$ to $\boldsymbol{R}_2\boldsymbol{x}$ to $\boldsymbol{R}_2^{-1}\boldsymbol{x}$ and back to $\boldsymbol{R}_1\boldsymbol{x}$, or $\boldsymbol{R}_1\boldsymbol{x}$ to $\boldsymbol{R}_2\boldsymbol{x}$ to $\boldsymbol{R}_2^{-1}\boldsymbol{x}$ to $\boldsymbol{R}_1^{-1}\boldsymbol{x}$ and back to $\boldsymbol{R}_1\boldsymbol{x}$, or $\boldsymbol{R}_1\boldsymbol{x}$ to $\boldsymbol{R}_2\boldsymbol{x}$ to $\boldsymbol{R}_1^{-1}\boldsymbol{x}$ to $\boldsymbol{R}_2^{-1}\boldsymbol{x}$ and back to $\boldsymbol{R}_1\boldsymbol{x}$. We consider these three cases separately.

Case 1.1

The vertex figure of $\boldsymbol{x}$ contains $\boldsymbol{R}_1\boldsymbol{x}$, $\boldsymbol{R}_1^{-1}\boldsymbol{x}$, $\boldsymbol{R}_2\boldsymbol{x}$, $\boldsymbol{R}_2^{-1}\boldsymbol{x}$ in that order.

Into this vertex figure (figure 18*a*) we add the vertex $\boldsymbol{x}$ (which is actually out of the plane), and set up the vertex angles α, β, γ, δ of the faces meeting at $\boldsymbol{x}$. Although the four points are drawn in cyclic order in figure 18*a*, we may include any non-cyclic order by allowing some of α, β, γ, δ to be negative. Also, there is no assumption that the four faces surround the direction of $\boldsymbol{x}$ once. They could equally well not surround $\boldsymbol{x}$ at all, or indeed could surround $\boldsymbol{x}$ in the reverse sense, without invalidating the following arguments.

We now begin building up the pattern of edges and faces round the vertex $\boldsymbol{x}$ by considering the vertex figures of neighbouring vertices. Consider the vertex figure of $\boldsymbol{R}_1\boldsymbol{x}$, obtained by applying $\boldsymbol{R}_1$ to the vertex figure of $\boldsymbol{x}$. If $\boldsymbol{R}_1$ is a proper rotation, the angles α, β, γ, δ remain in the same sense in the new vertex figure. If, however, $\boldsymbol{R}_1$ is an improper rotation, then the reflexion operation implicit in $\boldsymbol{R}_1$ reverses the sense of these angles. The two possibilities are shown in figures 18*b*, *c* respectively. The possible vertex figures of $\boldsymbol{R}_1^{-1}\boldsymbol{x}$, $\boldsymbol{R}_2\boldsymbol{x}$ and $\boldsymbol{R}_2^{-1}\boldsymbol{x}$ are obtained similarly (figures 18*d*–*i*).

It is convenient to consider separately the cases when $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ represent proper, and improper, rotations.

Case 1.1.1. $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are both proper rotations.

The vertex figures 18*a*, *b*, *d*, *f*, *h* are used to generate the pattern of edges and faces near the vertex $\boldsymbol{x}$ (figure 19*a*). In a polyhedron with few vertices, or with triangular or square faces, some of the vertices shown may coalesce – this does not affect the arguments. In passing, we can see that $\beta = \delta$, since the faces F_2 and F_4 are each regular polygons, and hence F_2 and F_4 are polygons of the same type.

We can also see that the operator $\boldsymbol{R}_1\boldsymbol{R}_2$ (a proper rotation) takes three successive vertices of the face F_4 into their next-but-one neighbours. It follows that the plane of F_4 must be normal to the rotation axis of $\boldsymbol{R}_1\boldsymbol{R}_2$. *Successive* vertices of F_4 are accordingly generated by an operator which we may call $(\boldsymbol{R}_1\boldsymbol{R}_2)^{\frac{1}{2}}$, denoting a proper rotation about the axis of $\boldsymbol{R}_1\boldsymbol{R}_2$ through an angle θ of (modulo π) half the angle of rotation of $\boldsymbol{R}_1\boldsymbol{R}_2$ itself. (We return to the ambiguity in θ presently.) In particular, $(\boldsymbol{R}_1\boldsymbol{R}_2)^{\frac{1}{2}}$ takes $\boldsymbol{x}$ to $\boldsymbol{R}_1\boldsymbol{x}$, which is in the vertex figure.

Thus *we have found a third operator which satisfies equation* (3). The new operator $(\boldsymbol{R}_1\boldsymbol{R}_2)^{\frac{1}{2}}$ is independent of $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ because these operators are independent of each other. Similarly, by considering the face F_2 we may show that the operator $(\boldsymbol{R}_2\boldsymbol{R}_1)^{\frac{1}{2}}$ also satisfies equation (3) – note that $\boldsymbol{R}_2\boldsymbol{R}_1$ is of the same type as $\boldsymbol{R}_1\boldsymbol{R}_2$ and also represents a proper rotation.

We may cover all the polyhedra contained in case 1.1.1 by enumerating the relatively small number of suitable *pairs* of independent operators of the group, and then using any convenient three of the operators $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, $(\boldsymbol{R}_1\boldsymbol{R}_2)^{\frac{1}{2}}$, $(\boldsymbol{R}_2\boldsymbol{R}_1)^{\frac{1}{2}}$ to determine $\boldsymbol{x}$.

For full generality, we should include both possible definitions of $(\boldsymbol{R}_1\boldsymbol{R}_2)^{\frac{1}{2}}$, which definitions differ by π in the angle of rotation. However, when $\boldsymbol{R}_1\boldsymbol{R}_2$ is a 3-, 5-, or $\frac{5}{2}$-fold rotation, one of these

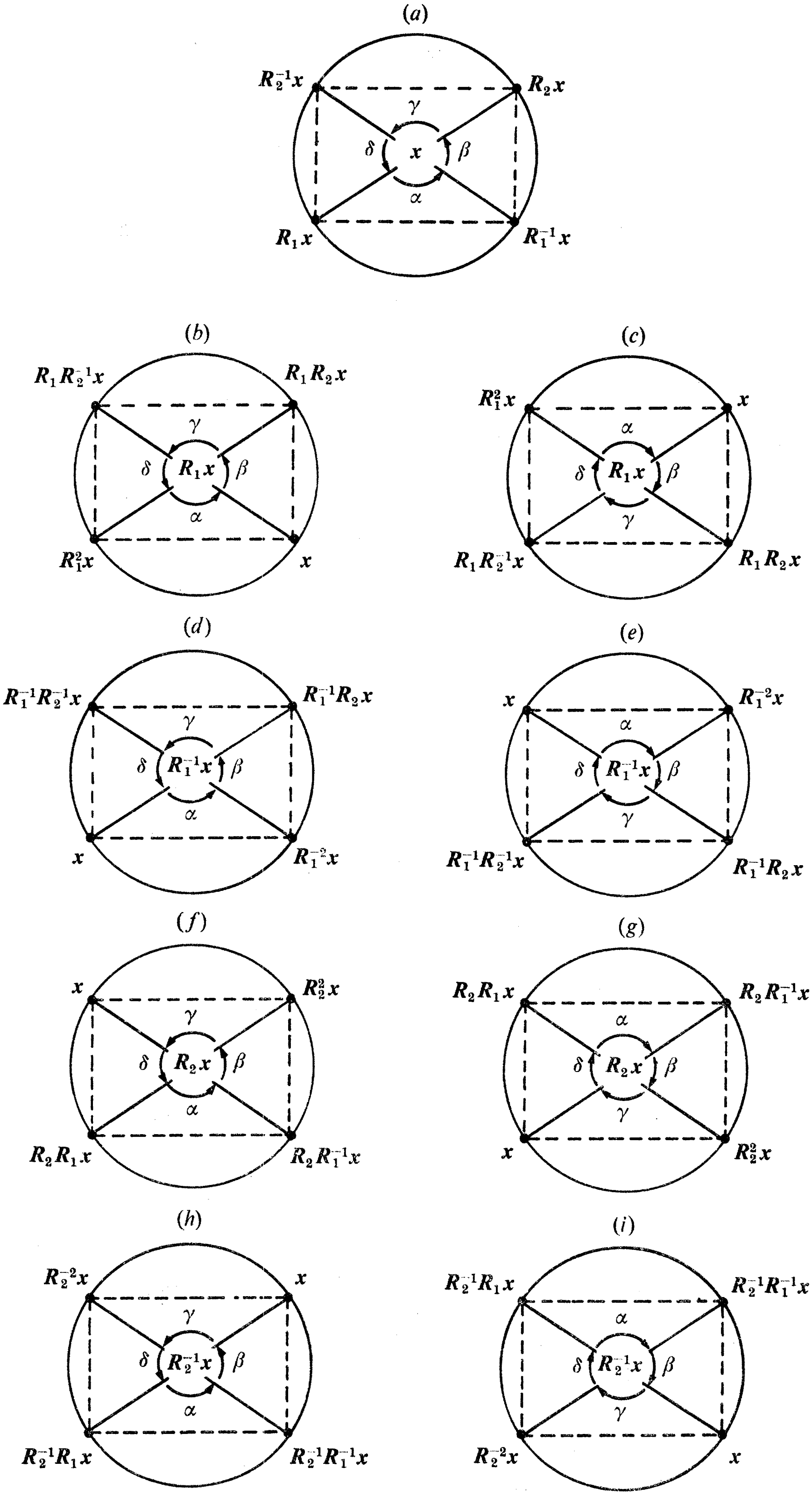

FIGURE 18. (*a*) Vertex figure of $\boldsymbol{x}$ (case 1.1). (*b*) Vertex figure of $\boldsymbol{R}_1\boldsymbol{x}$ when $\boldsymbol{R}_1$ proper. (*c*) Vertex figure of $\boldsymbol{R}_1\boldsymbol{x}$ when $\boldsymbol{R}_1$ improper. (*d*) Vertex figure of $\boldsymbol{R}_1^{-1}\boldsymbol{x}$ when $\boldsymbol{R}_1$ proper. (*e*) Vertex figure of $\boldsymbol{R}_1^{-1}\boldsymbol{x}$ when $\boldsymbol{R}_1$ improper. (*f*) vertex figure of $\boldsymbol{R}_2\boldsymbol{x}$ when $\boldsymbol{R}_2$ proper. (*g*) Vertex figure of $\boldsymbol{R}_2\boldsymbol{x}$ when $\boldsymbol{R}_2$ improper. (*h*) Vertex figure of $\boldsymbol{R}_2^{-1}\boldsymbol{x}$ when $\boldsymbol{R}_2$ proper. (*i*) Vertex figure of $\boldsymbol{R}_2^{-1}\boldsymbol{x}$ when $\boldsymbol{R}_2$ improper.

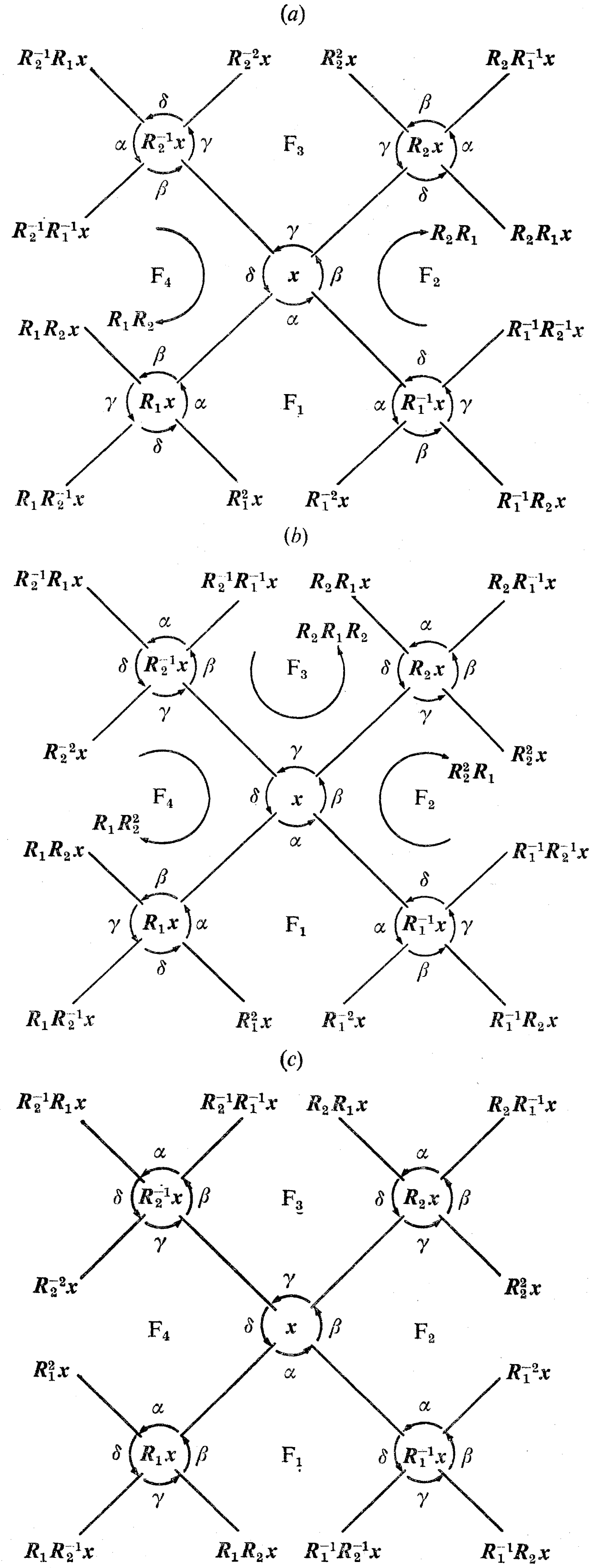

FIGURE 19 (*a*) Edges and faces near x when R_1 and R_2 are both proper (case 1.1.1). (*b*) Edges and faces near x when R_1 is proper and R_2 improper (case 1.1.2). (*c*) Edges and faces near x when R_1 and R_2 are both improper (case 1.1.4).

definitions of $(\boldsymbol{R}_1\boldsymbol{R}_2)^{\frac{1}{2}}$ coincides with an integer power of $\boldsymbol{R}_1\boldsymbol{R}_2$, which is an element of the symmetry group. These cases will already have been covered by the lists of three operators belonging to the group. When $\boldsymbol{R}_1\boldsymbol{R}_2$ is a fourfold rotation, both definitions of $(\boldsymbol{R}_1\boldsymbol{R}_2)^{\frac{1}{2}}$ must be used. Lastly, when $\boldsymbol{R}_1\boldsymbol{R}_2$ is a twofold rotation, both definitions of $(\boldsymbol{R}_1\boldsymbol{R}_2)^{\frac{1}{2}}$, having $\theta = \pm\frac{1}{2}\pi$, should be used (unless the rotation is about a fourfold axis, in which case both definitions coincide with elements of the group). However, since the computations only use $\cos\theta$ (see equation (3)), the sign of the rotation angle is irrelevant and the two definitions of $(\boldsymbol{R}_1\boldsymbol{R}_2)^{\frac{1}{2}}$ are computationally equivalent.

Case 1.1.2. $\boldsymbol{R}_1$ is a proper rotation, and $\boldsymbol{R}_2$ is improper.

The vertex figures 18*a*, *b*, *d*, *g*, *i* are used to generate the edges and faces near $\boldsymbol{x}$ (figure 19*b*). Note that the sense of the angles at $\boldsymbol{R}_2\boldsymbol{x}$ and $\boldsymbol{R}_2^{-1}\boldsymbol{x}$ has been reversed from that shown in figures 18*g*, *i*. This has been done to ensure that the edges of each face can be traversed serially from one to the next without reversing direction after each improper rotation, so that vertex angles within a face can be directly compared. By inspection of face F_3, $\beta = \gamma = \delta$, so that faces F_2, F_3, F_4 are all of the same type.

In face F_2, the proper rotation $\boldsymbol{R}_2^2\boldsymbol{R}_1$ takes each of two successive vertices into its next-but-two neighbour. The case when F_2 is a triangle is special, because $\boldsymbol{R}_2^2\boldsymbol{R}_1$ then takes each of the three vertices $\boldsymbol{x}$, $\boldsymbol{R}_2\boldsymbol{x}$, $\boldsymbol{R}_1^{-1}\boldsymbol{x}$ into itself. The vertex figure then degenerates (because these three points all lie along the axis of $\boldsymbol{R}_2^2\boldsymbol{R}_1$) unless $\boldsymbol{R}_2^2\boldsymbol{R}_1$ is the identity operator. In this latter case, with $\boldsymbol{R}_1 = \boldsymbol{R}_2^{-2}$, the polyhedron is an *antiprism*, with triangular side-faces.

When F_2 is not a triangle, $\boldsymbol{R}_2^2\boldsymbol{R}_1$ cannot be the identity, and the plane of F_2 must be normal to the rotation axis of $\boldsymbol{R}_2^2\boldsymbol{R}_1$. Similarly, F_3 is normal to the rotation axis of $\boldsymbol{R}_2\boldsymbol{R}_1\boldsymbol{R}_2$ and F_4 is normal to the rotation axis of $\boldsymbol{R}_1\boldsymbol{R}_2^2$. Since F_2, F_3, F_4 are faces of the same type, $\boldsymbol{x}$ (which is a vertex of each) must be equidistant from the axes of $\boldsymbol{R}_2^2\boldsymbol{R}_1$, $\boldsymbol{R}_2\boldsymbol{R}_1\boldsymbol{R}_2$, $\boldsymbol{R}_1\boldsymbol{R}_2^2$, which axes are themselves related through successive rotations by $\boldsymbol{R}_2$ (less inversion). To be more precise, since these three operators rotate F_3 in the opposite sense from F_2 and F_4, the components of $\boldsymbol{x}$ resolved along the axes of $\boldsymbol{R}_2^2\boldsymbol{R}_1$ and $\boldsymbol{R}_1\boldsymbol{R}_2^2$ are equal to *minus* the component of $\boldsymbol{x}$ along the axis of $\boldsymbol{R}_2\boldsymbol{R}_1\boldsymbol{R}_2$ (defining the senses of these axes in a consistent way by the rotation $\boldsymbol{R}_2$, less inversion, between them). A straightforward investigation of all the independent pairs of operators $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ shows that in each case, any $\boldsymbol{x}$ which is equidistant from the three axes just described must lie on a symmetry plane or symmetry axis. Thus there is always some symmetry operation, $\boldsymbol{R}_3$ say, which takes $\boldsymbol{x}$ into itself. Then $\boldsymbol{R}_1\boldsymbol{x}$ (for example) may also be described as $\boldsymbol{R}_1\boldsymbol{R}_3\boldsymbol{x}$, and we have a third member of the group, namely $\boldsymbol{R}_1\boldsymbol{R}_3$, satisfying equation (3). (We can always find $\boldsymbol{R}_3$ such that $\boldsymbol{R}_1\boldsymbol{R}_3$ differs from $\boldsymbol{R}_1^{\pm 1}$ and $\boldsymbol{R}_2^{\pm 1}$.)

Accordingly, antiprisms apart, this case will have been covered by the lists of three operators of the group.

Case 1.1.3. $\boldsymbol{R}_1$ is an improper rotation, and $\boldsymbol{R}_2$ is proper.

This reduces to case 1.1.2 by relabelling $\boldsymbol{R}_1$ as $\boldsymbol{R}_2$ and vice versa, so may be discarded.

Case 1.1.4. $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are both improper rotations.

The vertex figures 18*a*, *c*, *e*, *g*, *i* are used to generate the edges and faces near $\boldsymbol{x}$ (figure 19*c*). Faces F_1 and F_3 show that $\alpha = \beta = \gamma = \delta$. Accordingly the vertex figure of $\boldsymbol{x}$ must be a *square* and the points $(\boldsymbol{R}_1\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{R}_2\boldsymbol{x})$ and $(\boldsymbol{R}_1^{-1}\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{R}_2^{-1}\boldsymbol{x})$ lie on orthogonal great circles. The former great circle also passes through $R_1R_2^{-1}x$ and $R_2R_1^{-1}x$ (figure 19*c*), giving us five equally spaced points along it, namely $R_1R_2^{-1}x$, R_1x, x, R_2x, $R_2R_1^{-1}x$. Now the operator $R_2R_1^{-1}$, which represents a proper rotation, takes each vertex of this list into its next-but-one neighbour, so that the great circle lies normal to the axis of $R_2R_1^{-1}$. In particular, x lies normal to the axis of $R_2R_1^{-1}$. Similarly, by considering the other great circle, x lies normal to the axis of $R_1^{-1}R_2$. Furthermore, since the great circles are orthogonal, the axes of $R_2R_1^{-1}$ and $R_1^{-1}R_2$ (which represent proper rotations of the same type) must be orthogonal to each other.

These orthogonality conditions always force x onto a twofold or fourfold symmetry axis, so that a reflexion operator R_3 can be found which takes x into itself. Then R_1x may also be described as R_1R_3x, so that the symmetry operator R_1R_3 also satisfies equation (3). R_1R_3 clearly differs from $R_1^{\pm 1}$ and $R_2^{\pm 1}$ since it represents a proper rotation.

Accordingly this case will have been covered by the lists of three operators of the group.

Case 1.2

The vertex figure of x contains R_1x, R_2x, $R_2^{-1}x$, $R_1^{-1}x$ in that order.

This reduces to case 1.1 by relabelling R_1 as R_1^{-1}, so yields nothing new.

Case 1.3

The vertex figure of x contains R_1x, R_2x, $R_1^{-1}x$, $R_2^{-1}x$ in that order.

Into this vertex figure (figure 20) we add the vertex x and the vertex angles α, β, γ, δ. As before, there is no assumption that all of α, β, γ, δ are positive. The vertex figures of R_1x, R_2x, $R_1^{-1}x$, $R_2^{-1}x$ (analogous to figures 18*b*–*i*) are now used to generate the faces and edges near x. As before, we consider separately the cases when R_1 and R_2 represent proper, and improper, rotations.

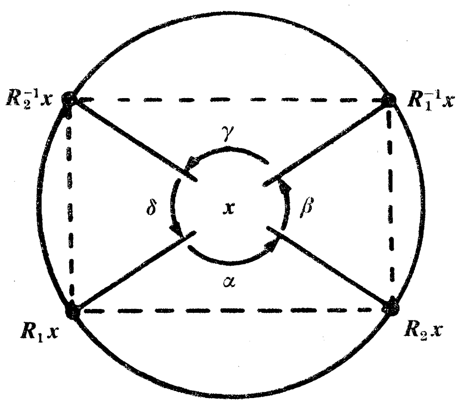

FIGURE 20. Vertex figure of x (case 1.3).

Case 1.3.1. R_1 and R_2 are both proper rotations.

This case gives the pattern of edges and faces shown in figure 21*a*. Faces F_1 and F_3 show that $\alpha = \beta = \gamma = \delta$. Hence the vertex figure of x must be a *square*, and the points $(R_1x,\ x,\ R_1^{-1}x)$ and $(R_2x,\ x,\ R_2^{-1}x)$ must lie on orthogonal great circles. Thus the direction of x is normal to the rotation axes of both R_1 and R_2.

Furthermore, since the edges (x, R_1x) and (x, R_2x) must be of equal length, the operators R_1 and R_2 must represent rotations of equal magnitude. The axes of R_1 and R_2 are orthogonal,

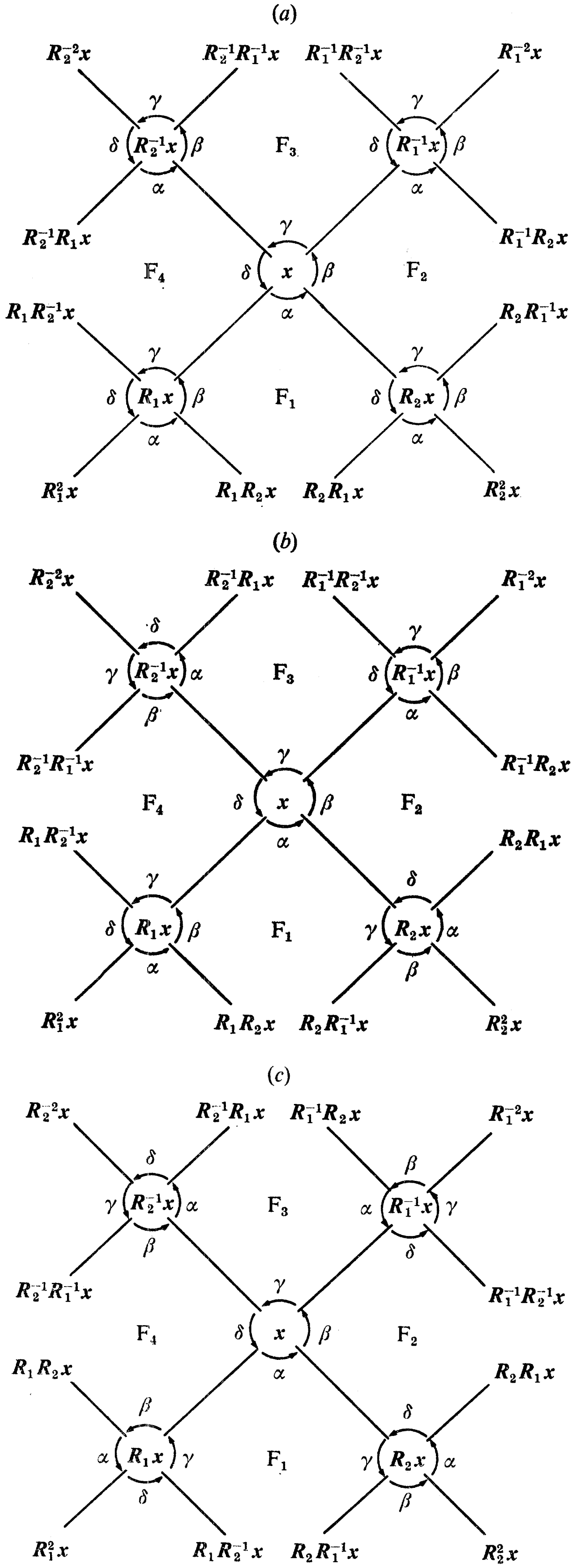

FIGURE 21. (a) Edges and faces near x when R_1 and R_2 are both proper (case 1.3.1). (b) Edges and faces near x when R_1 is proper and R_2 improper (case 1.3.2). (c) Edges and faces near x when R_1 and R_2 are both improper (case 1.3.4).

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

because the great circles are orthogonal. As explained under case 1.1.4, these orthogonality conditions allow us to generate further symmetry operators satisfying equation (3).

Accordingly this case will have been covered by the lists of three operators of the group.

Case 1.3.2. $\boldsymbol{R}_1$ is a proper rotation, and $\boldsymbol{R}_2$ is improper.

This case gives the pattern of edges and faces shown in figure 21 *b*. Faces F_1 and F_3 show that $\alpha = \beta = \gamma = \delta$. Accordingly the vertex figure is a *square* and the points $(\boldsymbol{R}_1^{-2}\boldsymbol{x}, \boldsymbol{R}_1^{-1}\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{R}_1\boldsymbol{x}, \boldsymbol{R}_1^2\boldsymbol{x})$ lie equally spaced along a great circle normal to the rotation axis of $\boldsymbol{R}_1$, while the points $(\boldsymbol{R}_2^{-2}\boldsymbol{x}, \boldsymbol{R}_2^{-1}\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{R}_2\boldsymbol{x}, \boldsymbol{R}_2^2\boldsymbol{x})$ lie equally spaced along an orthogonal great circle normal to the rotation axis of $\boldsymbol{R}_2$ (cf. the discussion of case 1.1.4).

The points along the former great circle form a 3-, 4-, 5- or $\frac{5}{2}$-gon, according to whether $\boldsymbol{R}_1$ represents a (proper) 3-, 4-, 5- or $\frac{5}{2}$-fold rotation. Similarly, the points along the latter great circle form a 6-, 4-, $\frac{10}{3}$- or 10-gon, according to whether $\boldsymbol{R}_2$ represents a (improper) 3-, 4-, 5- or $\frac{5}{2}$-fold rotation. Regular polygons inscribed in great circles must be of the same type, so they must both be 4-gons (squares). This forces $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ to be orthogonal (cubic) fourfold rotations, and $\boldsymbol{x}$ then lies along the remaining fourfold axis, giving in fact an octahedron.

Several other operators then satisfy equation (3), and this case is covered by the lists of three operators of the group.

Case 1.3.3. $\boldsymbol{R}_1$ is an improper rotation, and $\boldsymbol{R}_2$ is proper.

This reduces to case 1.3.2 by relabelling $\boldsymbol{R}_1$ as $\boldsymbol{R}_2$ and vice versa, so may be discarded.

Case 1.3.4. $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are both improper rotations.

This case gives the pattern of edges and faces shown in figure 21 *c*. Faces F_1 and F_2 show that $\alpha = \gamma$ and $\beta = \delta$. Hence the vertex figure of $\boldsymbol{x}$ must be a *rectangle*, and the points $(\boldsymbol{R}_1^{-1}\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{R}_1\boldsymbol{x})$ and $(\boldsymbol{R}_2^{-1}\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{R}_2\boldsymbol{x})$ must each lie on a great circle. This ensures that the direction of $\boldsymbol{x}$ is normal to the axes of both $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$. Furthermore, since the edges $(\boldsymbol{x}, \boldsymbol{R}_1\boldsymbol{x})$ and $(\boldsymbol{x}, \boldsymbol{R}_2\boldsymbol{x})$ must be of equal length, the operators $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ must represent rotations of equal magnitude.

However, if $\boldsymbol{x}$ is to be normal to two (non 2-fold but not necessarily orthogonal) symmetry axes of the same type, then $\boldsymbol{x}$ is always constrained to lie on a symmetry plane or symmetry axis. Hence there exists a reflexion operator $\boldsymbol{R}_3$ which leaves $\boldsymbol{x}$ unchanged, and we can construct $\boldsymbol{R}_1\boldsymbol{R}_3$ as a third member of the group satisfying equation (3). $\boldsymbol{R}_1\boldsymbol{R}_3$ clearly differs from $\boldsymbol{R}_1^{\pm 1}$ and $\boldsymbol{R}_2^{\pm 1}$ since it represents a proper rotation.

Accordingly this case will have been covered by the lists of three operators of the group.

Case 2

The vertex figure of $\boldsymbol{x}$ contains three points, namely $\boldsymbol{R}_1\boldsymbol{x}$, $\boldsymbol{R}_1^{-1}\boldsymbol{x}$, $\boldsymbol{R}_2\boldsymbol{x}$.

$\boldsymbol{R}_1$ must not be a twofold operator, otherwise $\boldsymbol{R}_1\boldsymbol{x}$ and $\boldsymbol{R}_1^{-1}\boldsymbol{x}$ will coalesce. There is now only one way to join up the vertex figure (figure 22 *a*). Since $\boldsymbol{x}$ is joined to each of $(\boldsymbol{R}_1\boldsymbol{x}, \boldsymbol{R}_1^{-1}\boldsymbol{x}, \boldsymbol{R}_2\boldsymbol{x})$, $\boldsymbol{R}_2\boldsymbol{x}$ must be joined to each of $(\boldsymbol{R}_2\boldsymbol{R}_1\boldsymbol{x}, \boldsymbol{R}_2\boldsymbol{R}_1^{-1}\boldsymbol{x}, \boldsymbol{R}_2^2\boldsymbol{x})$. One of these latter points must be identified with $\boldsymbol{x}$.

Suppose that $\boldsymbol{R}_2\boldsymbol{R}_1\boldsymbol{x} = \boldsymbol{x}$. Then $\boldsymbol{R}_1^{-1}\boldsymbol{x}$ could equally well be written as $\boldsymbol{R}_1^{-1}\boldsymbol{R}_2\boldsymbol{R}_1\boldsymbol{x}$. Now $\boldsymbol{R}_1^{-1}\boldsymbol{R}_2\boldsymbol{R}_1$ is an operator of the same type as $\boldsymbol{R}_2$, but about a different axis. Hence $\boldsymbol{R}_1^{-1}\boldsymbol{R}_2\boldsymbol{R}_1$ is a third operator of the group (not equal to $\boldsymbol{R}_2$ or $\boldsymbol{R}_1^{\pm 1}$ otherwise the vertex figure degenerates) which takes $\boldsymbol{x}$ into a point in its vertex figure, and hence satisfies equation (3). This will have been covered by the lists of three operators, so we may discard the possibility that $\boldsymbol{R}_2\boldsymbol{R}_1\boldsymbol{x} = \boldsymbol{x}$. Similarly, we discard the possibility that $\boldsymbol{R}_2\boldsymbol{R}_1^{-1}\boldsymbol{x} = \boldsymbol{x}$.

The only remaining possibility is $\boldsymbol{R}_2^2\boldsymbol{x} = \boldsymbol{x}$, which, since $\boldsymbol{x}$ must not lie along the axis of $\boldsymbol{R}_2$ (otherwise the vertex figure degenerates), shows that $\boldsymbol{R}_2$ *must be a* (proper or improper) *twofold operator*, with $\boldsymbol{R}_2^2$ being the identity operator.

We now construct the possible vertex figures of $\boldsymbol{R}_1\boldsymbol{x}$, $\boldsymbol{R}_1^{-1}\boldsymbol{x}$ and $\boldsymbol{R}_2\boldsymbol{x}$ (figures 22 *b*, *c*, *d*, *e*, *f*, *g*), and consider separately the cases when $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ represent proper, and improper, rotations.

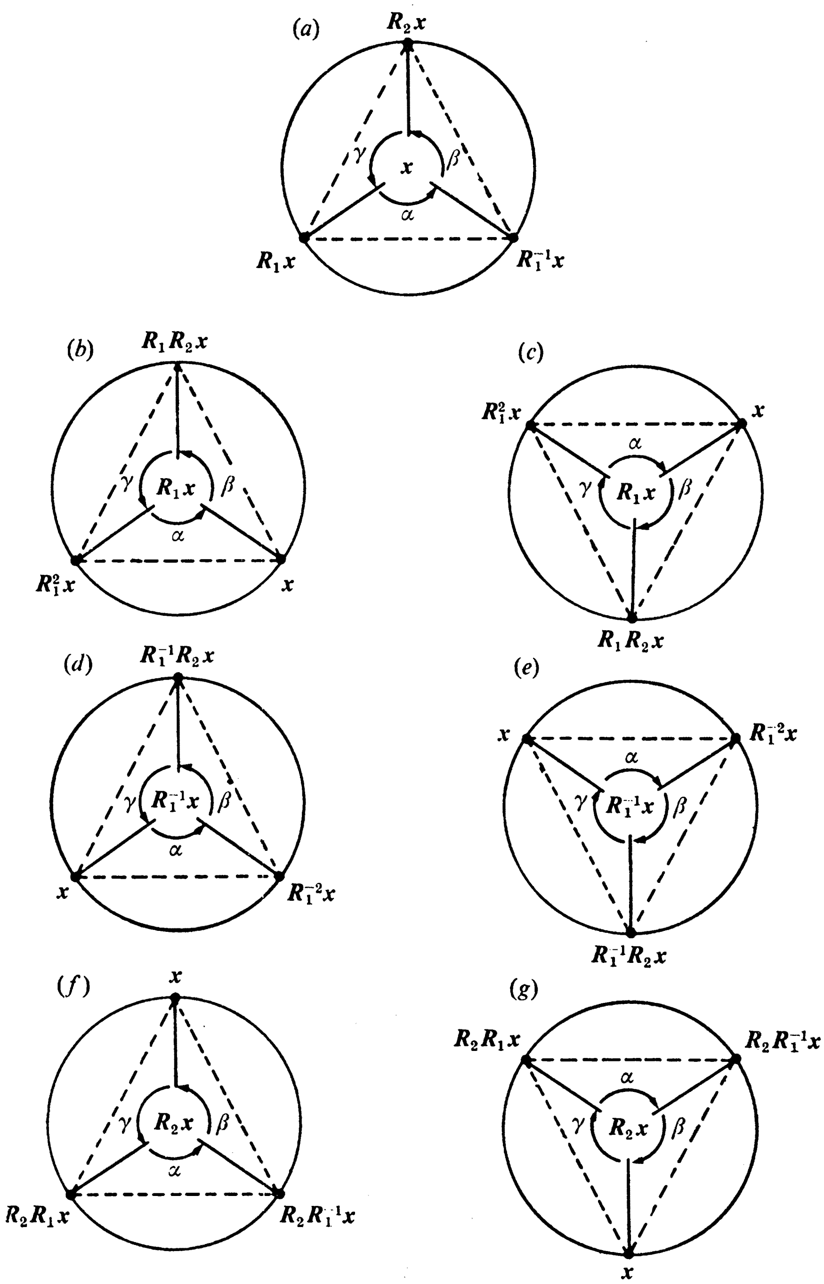

FIGURE 22. (*a*) Vertex figure of $\boldsymbol{x}$ (case 2). (*b*) Vertex figure of $\boldsymbol{R}_1\boldsymbol{x}$ when $\boldsymbol{R}_1$ proper. (*c*) Vertex figure of $\boldsymbol{R}_1\boldsymbol{x}$ when $\boldsymbol{R}_1$ improper. (*d*) Vertex figure of $\boldsymbol{R}_1^{-1}\boldsymbol{x}$ when $\boldsymbol{R}_1$ proper. (*e*) Vertex figure of $\boldsymbol{R}_1^{-1}\boldsymbol{x}$ when $\boldsymbol{R}_1$ improper. (*f*) Vertex figure of $\boldsymbol{R}_2\boldsymbol{x}$ when $\boldsymbol{R}_2$ proper. (*g*) Vertex figure of $\boldsymbol{R}_2\boldsymbol{x}$ when $\boldsymbol{R}_2$ improper.

Case 2.1

$\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are both proper rotations.

This case gives the pattern of edges and faces shown in figure 23*a*. Face F_3 shows that $\beta = \gamma$, so that F_2 and F_3 are faces of the same type.

We can also see that the operator $\boldsymbol{R}_1\boldsymbol{R}_2$ (a proper rotation) takes three successive vertices of the face F_3 into their next-but-one neighbours, and $\boldsymbol{R}_2\boldsymbol{R}_1$ does the same for F_2. It follows that we can deal with this case by defining a new operator $(\boldsymbol{R}_1\boldsymbol{R}_2)^{\frac{1}{2}}$, or alternatively $(\boldsymbol{R}_2\boldsymbol{R}_1)^{\frac{1}{2}}$, as explained in the discussion of case 1.1.1.

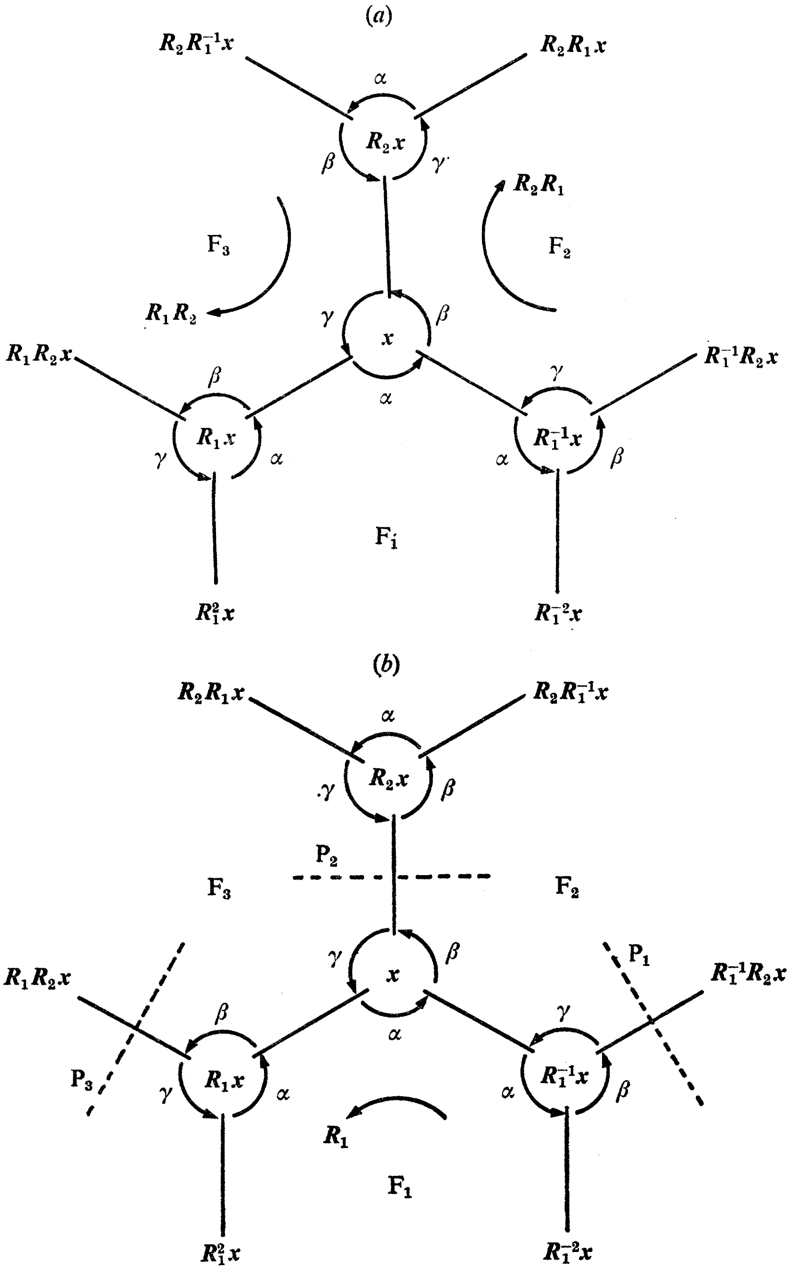

FIGURE 23. For description see opposite.

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A

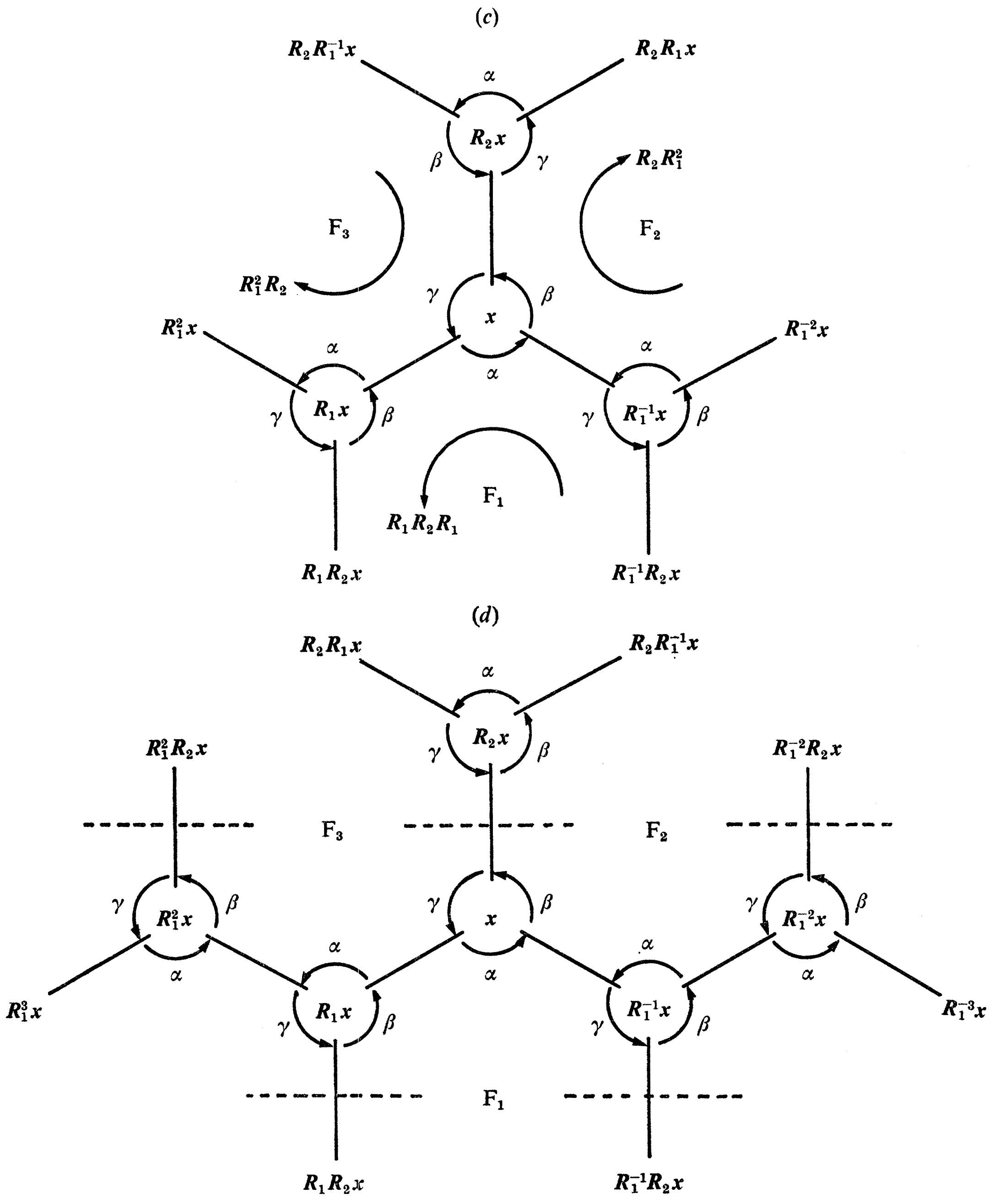

FIGURE 23. (*a*) Edges and faces near x when R_1 and R_2 are both proper (case 2.1). (*b*) Edges and faces near x when R_1 is proper and R_2 improper (case 2.2). (*c*) Edges and faces near x when R_1 is improper and R_2 proper (case 2.3). (*d*) Edges and faces near x when R_1 and R_2 are both improper (case 2.4).

Case 2.2

R_1 is a proper rotation, and R_2 is improper.

This case gives the pattern of edges and faces shown in figure 23*b*. Face F_3 shows that $\beta = \gamma$, so that F_2 and F_3 are faces of the same type.

Now the perpendicular bisector plane P_2 of the edge (x, R_2x), shown as a dashed line in figure 23*b*, is the reflexion plane of R_2, and this plane must pass through the centres of faces

F_2 and F_3. Similarly, the bisector plane P_3 of the edge $(\boldsymbol{R}_1\boldsymbol{x}, \boldsymbol{R}_1\boldsymbol{R}_2\boldsymbol{x})$ is the reflexion plane of $\boldsymbol{R}_1\boldsymbol{R}_2\boldsymbol{R}_1^{-1}$, and this plane too must pass through the centre of F_3. Disregarding the case when F_2 and F_3 are squares (which merely gives a prism with square side-faces), this means that the centre of F_3 lies at the intersection of two different symmetry planes P_2 and P_3, which are related by a rotation $\boldsymbol{R}_1$. Hence F_3 is centred on a symmetry axis. Similarly, F_2 is centred on the symmetry axis defined by the intersection of planes P_1 and P_2, which are also related by a rotation $\boldsymbol{R}_1$. These two symmetry axes are clearly related by a rotation of $\boldsymbol{R}_1$, and hence are of the same type.

Thus the centres of F_2 and F_3 lie on symmetry axes of the same type, both lying on a common symmetry plane P_2. This always forces $\boldsymbol{x}$, which lies on the perpendicular bisector plane of the line joining the centres of F_2 and F_3, to lie on a symmetry plane of the group: the degenerate case when the centres of F_2 and F_3 coincide at the polyhedron centre is discarded because the polyhedron then collapses to a plane. The reflexion operator $\boldsymbol{R}_3$ which takes $\boldsymbol{x}$ into itself can be used to generate a third symmetry operator $\boldsymbol{R}_1\boldsymbol{R}_3$ (different from $\boldsymbol{R}_2$ and $\boldsymbol{R}_1^{\pm 1}$) which satisfies equation (3).

Accordingly, prisms apart, this case will have been covered by the lists of three operators of the group.

Case 2.3

$\boldsymbol{R}_1$ is an improper rotation, and $\boldsymbol{R}_2$ is proper.

This case gives the pattern of edges and faces shown in figure 23*c*. Face F_1 shows that $\alpha = \beta = \gamma$, so that all the faces are of the same type.

The proper rotation $\boldsymbol{R}_1^2\boldsymbol{R}_2$ takes each of two successive vertices of F_3 into its next-but-two neighbour, and $\boldsymbol{R}_2\boldsymbol{R}_1^2$ does the same for face F_2. $\boldsymbol{R}_1\boldsymbol{R}_2\boldsymbol{R}_1$ does the same for face F_1, though the rotation is then in the opposite sense. As in case 1.1.2, the case when the faces are triangles is special. For a non-degenerate vertex figure in this special case, $\boldsymbol{R}_2$ must be identified with $\boldsymbol{R}_1^{-2}$. Since $\boldsymbol{R}_2$ is twofold, $\boldsymbol{R}_1$ is forced to be a (improper) fourfold rotation. The polyhedron is a *tetrahedron* (i.e. an antiprism with digon end-faces), with two opposite edges bisected by the fourfold axis.

When the faces are not triangles (cf. the discussion of case 1.1.2) the components of $\boldsymbol{x}$ resolved along the axes of $\boldsymbol{R}_1^2\boldsymbol{R}_2$ and $\boldsymbol{R}_2\boldsymbol{R}_1^2$ are equal to minus the component of $\boldsymbol{x}$ along the axis of $\boldsymbol{R}_1\boldsymbol{R}_2\boldsymbol{R}_1$ (defining the senses of the axes in a consistent way by the rotation of $\boldsymbol{R}_1$, less inversion, between them). A straightforward investigation of the possible cases shows that such an $\boldsymbol{x}$ always lies on a symmetry plane. The corresponding reflexion operator can then be used to construct a third symmetry operator satisfying equation (3), as in case 1.1.2.

Accordingly, the tetrahedron apart, this case will have been covered by the lists of three operators of the group.

Case 2.4

$\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are both improper rotations.

This case gives the pattern of edges and faces shown in figure 23*d*. Face F_1 shows that $\alpha = \beta = \gamma$, so that all the faces are of the same type.

Consider face F_1. The perpendicular bisector plane of the edge $(\boldsymbol{R}_1\boldsymbol{x}, \boldsymbol{R}_1\boldsymbol{R}_2\boldsymbol{x})$ is the reflexion plane of $\boldsymbol{R}_1\boldsymbol{R}_2\boldsymbol{R}_1^{-1}$, and the perpendicular bisector plane of the next-but-two edge $(\boldsymbol{R}_1^{-1}\boldsymbol{x}, \boldsymbol{R}_1^{-1}\boldsymbol{R}_2\boldsymbol{x})$ is the reflexion plane of $\boldsymbol{R}_1^{-1}\boldsymbol{R}_2\boldsymbol{R}_1$. Hence, except when F_1 is a triangle or hexagon, the centre of F_1 lies at the intersection of at least two symmetry planes. Thus F_1 must be centred on a symmetry axis. (When F_1 is a triangle, the polyhedron is simply a tetrahedron arranged symmetrically about reflexion planes, and this case is covered by the lists of three operators. When F_1 is a hexagon, there is no polyhedron at all because hexagons, meeting three to a vertex, tessellate a plane.)

Similarly, F_3 is centred on the intersection of the symmetry planes which bisect the edge $(\boldsymbol{x}, \boldsymbol{R}_2\boldsymbol{x})$ and the next-but-two edge $(\boldsymbol{R}_1^2\boldsymbol{x}, \boldsymbol{R}_1^2\boldsymbol{R}_2\boldsymbol{x})$, while F_2 is centred on the intersection of the symmetry planes which bisect the edge $(\boldsymbol{R}_1^{-2}\boldsymbol{x}, \boldsymbol{R}_1^{-2}\boldsymbol{R}_2\boldsymbol{x})$ and the next-but-two edge $(\boldsymbol{x}, \boldsymbol{R}_2\boldsymbol{x})$. Since F_1, F_2, F_3 are faces of the same type, these pairs of symmetry planes intersect at the same angle, and the symmetry axes through their centres must then be of the same type also (save for an ambiguity between twofold and fourfold cubic axes if the planes are orthogonal). Furthermore, the three symmetry axes are equidistant from each other, and also equidistant from $\boldsymbol{x}$. These conditions force $\boldsymbol{x}$ to lie on a symmetry plane or symmetry axis. The reflexion operator $\boldsymbol{R}_3$ which takes $\boldsymbol{x}$ into itself can be used to generate a third symmetry operator $\boldsymbol{R}_1\boldsymbol{R}_3$ (different from $\boldsymbol{R}_2$ and $\boldsymbol{R}_1^{\pm 1}$ because it represents a proper rotation) which satisfies equation (3).

Accordingly, this final case will have been covered by the lists of three operators of the group.

To summarize, all polyhedra in which $\boldsymbol{x}$ is connected to other vertices by *only two* operators of the group can be found by enumerating the relatively small number of independent pairs $(\boldsymbol{R}_1, \boldsymbol{R}_2)$ of proper rotations in the group, forming either or both of $(\boldsymbol{R}_1\boldsymbol{R}_2)^{\frac{1}{2}}$ and $(\boldsymbol{R}_2\boldsymbol{R}_1)^{\frac{1}{2}}$, and then using any three of these operators to define the position of $\boldsymbol{x}$.

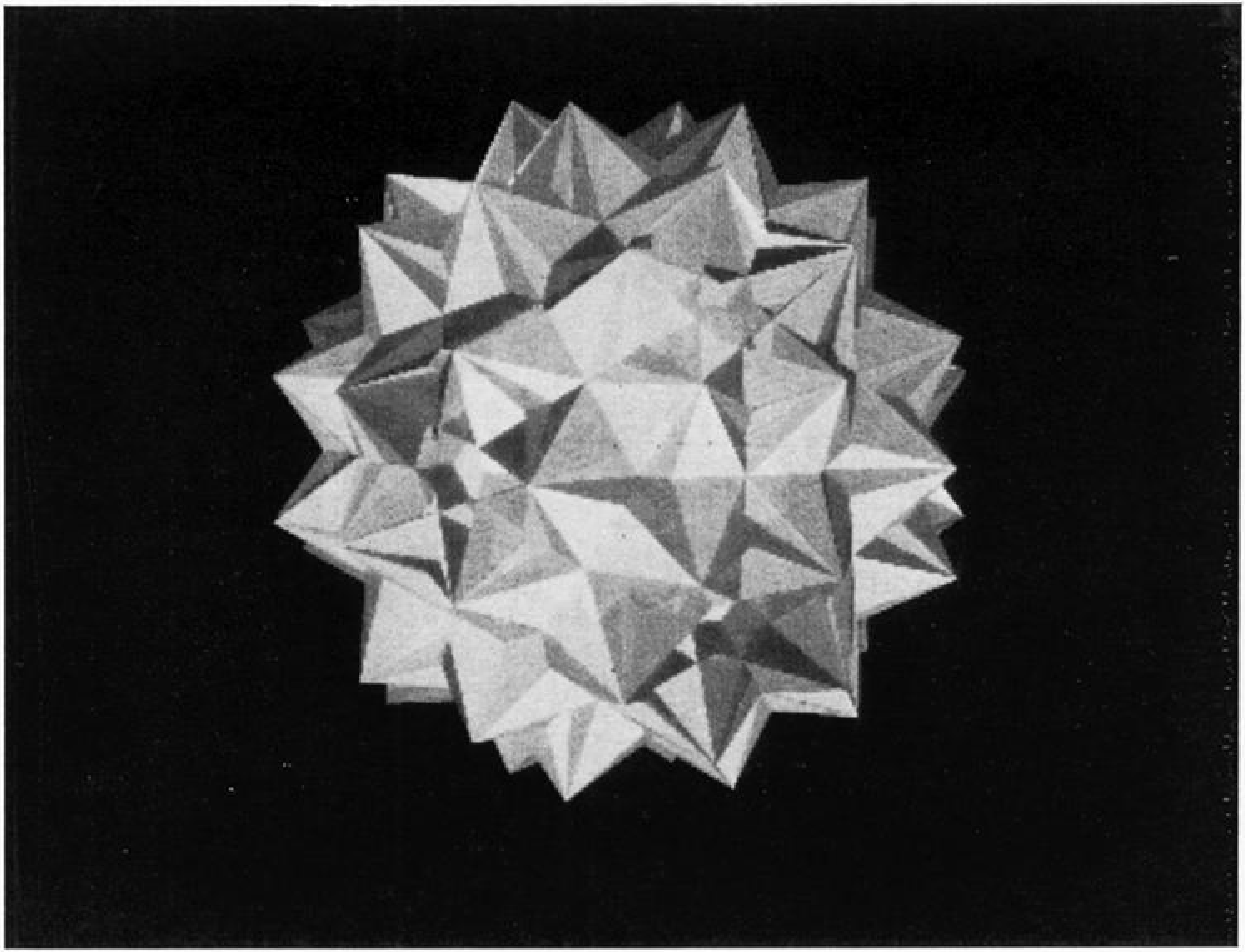

FIGURE 17. Computer photograph of great disnub dirhombidodecahedron.